**PACF for ARMA**

The ACF really helps to find the order of a MA(q) model, but does not help much for finding the order of an AR(p) (because the ACF does not “cut-off” to 0). The partial autocorrelation function (PACF) helps to find the order of an AR(p)!

Consider the following autoregressive representation:

 where wt+h ~ independent N(0,σ2)

NOTE: The different notation is used here to help motivate you to think of this as a regression model.

Remember the interpretation of a β parameter - Given all of the other variables in the model, β measures the relationship between its corresponding independent variable and the dependent variable.

Multiplying both sides by xt+h-j and taking the expectation produces:



Note that  = E(xt+h-j)E(wt+h) because xt+h-j and wt+h are independent (xt+h-j has w’s with subscripts lower than t+h). Then  = E(xt+h-j)×0 = 0.

Continuing,

 



Remember ρ(j) = γ(j)/γ(0).

Then for j = 1,…,h:



Remember that ρ(-j) = ρ(j)

Suppose h = 1, then

ρ(1) = β11ρ(0) ⇒ β11 = ρ(1) because ρ(0) = 1

Remember if h = 1, the model is 

Suppose h = 2, then

ρ(1) = β21ρ(0) + β22ρ(1)

ρ(2) = β21ρ(1) + β22ρ(0)

⇒ β22 =  from solving the above system of equations.

Suppose h = 3, then

ρ(1) = β31ρ(0) + β32ρ(1) + β33ρ(2)

ρ(2) = β31ρ(1) + β32ρ(0) + β33ρ(1)

ρ(3) = β31ρ(2) + β32ρ(1) + β33ρ(0)

⇒β33 = 

This process can be continued to find βhh.

These β’s are called partial autocorrelations because they measure the dependence of xt on xt+h removing the effect of all the other random variables in between. Thus,

β11 = Corr(xt, xt+1)

β22 = Corr(xt, xt+2 | xt+1) where “|” means “given”

β33 = Corr(xt, xt+3 | xt+1, xt+2)

Also, these can be treated like “regular” correlations in terms of scale: -1 ≤ βhh ≤ 1.

Most textbooks use ϕhh to denote the partial autocorrelations at lag h.

**THIS CAN BE VERY CONFUSING SINCE MOST BOOKS USE ϕj TO DENOTE A PARAMETER FROM AN AUTOREGRESSIVE MODEL!!!!**

I will use this notation from now on. Thus, βhh in the old notation is ϕhh in the new notation!!!

Example: AR(1) (ar1\_sim.R)

We have seen that ρ(h) = .

ϕ11 = ρ(1) = ϕ1

ϕ22 =  =  = 0

ϕ33 = 0,…

Thus, a AR(1)’s PACF cuts off to 0 after lag 1.

Suppose ϕ1 = 0.7. UsingARMAacf() with the PACF = TRUEoption produces

> ARMAacf(ar = c(0.7), lag.max = 20, pacf = TRUE)

[1] 0.7 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0

 0.0 0.0 0.0 0.0 0.0 0.0

> plot(x = ARMAacf(ar = c(0.7), lag.max = 20, pacf = TRUE),

 type = "h", ylim = c(-1,1), xlab = "h", ylab =

 expression(phi1[hh]), main = expression(paste(“PACF

 for AR(1) with “, phi1[1] == 0.7)))

> abline(h = 0)



Note that the first PACF value given is for h = 1 so the problems encountered earlier with the ACF and the plot() function do not occur here.

Example: AR(2)

ρ(h) = ϕ1ρ(h-1) + ϕ2ρ(h-2) and and  from past notes.

Then, ϕ11 = ρ(1) =  and





ϕ33 = 0,…

An AR(2)’s PACF cuts off to 0 after lag 2.

Notes:

* ARMA models with q > 0 do not have the partial correlations “cut-off” to 0. Instead, they behave like the ACF does for models with p > 0.
* Verify on your own that



for an MA(1) (see Shumway and Stoffer). From ma1\_sim.R (see code in program),



* Estimating the PACF



where . For more on this expression, see the exercises of Shumway and Stoffer. We will not calculate these by hand. Rather, we will use the pacf() function in R.

An approximation for the standard error is . To test

Ho:ϕhh = 0

Ha:ϕhh ≠ 0

we can check if  is within ±Z1-α/2 × n-1/2.

* Why would you want to do the above hypothesis test?
* Suppose you observed a time series. The estimated values of the PACF are  ≠ 0 and  ≈ 0 for h > 1. What may be a good model for the data?
* Suppose you observed a time series. The estimated values of the PACF are  ≠ 0,  ≠ 0, and  ≈ 0 for h > 2. What may be a good model for the data?
* Table of PACFs for common ARMA models

|  |  |  |
| --- | --- | --- |
| AR(1):xt=ϕ1xt-1+wt | ϕhh = |  |
| AR(2): xt=ϕ1xt-1+ϕ2xt-2+wt | ϕhh = | ϕ11=ρ(1)ϕ22=ϕ2ϕhh=0 for h3 |
| MA(1): xt = θ1wt-1 + wt | ϕhh = |  for h1  |
| MA(2): xt = θ1wt-1 +θ2wt-2+wt | ϕhh = |  |

Remember: “+” signs are used in the moving average operator, θ(B). **Many textbooks use “-“ signs so be careful when you examine these book!!!!** Thus, PACFs may be slightly different.

In summary,

|  |  |  |  |
| --- | --- | --- | --- |
|  | **AR(p)** | **MA(q)** | **ARMA(p,q)** |
| PACF | Cuts off to 0 after lag p | Tails off to 0 | Tails off to 0 after lag p |